Chapter 5: Numerical Methods in Heat Transfer

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Objectives

• Understand the limitations of analytical solutions of conduction problems, and the need for computation-intensive numerical methods,

• Express derivatives as differences, and obtain finite difference formulations,

• Solve steady one-dimensional conduction problems numerically using the finite difference method, and

• Solve transient one- or two-dimensional conduction problems using the finite difference method.
Why Numerical Methods?

1. Limitations  □  Analytical solution methods are limited to *highly simplified problems* in *simple geometries*.

2. Better Modeling  □  
   An “approximate” solution is usually more accurate than the “exact” solution of a crude mathematical model.

3. Flexibility  □  
   Engineering problems often require extensive *parametric studies*.

4. Complications  □  even when the analytical solutions are available, they might be quite intimidating.
Finite Difference Formulation

• The numerical methods for solving differential equations are based on replacing the differential equations by algebraic equations.

• For finite difference method, this is done by replacing the derivatives by differences.

• A function $f$ that depends on $x$.

• The first derivative of $f(x)$ at a point is equivalent to the slope of a line tangent to the curve at that point.
\[
\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

(5-5)
• If we don’t take the indicated limit, we will have the following approximate relation for the derivative:

\[
\frac{df(x)}{dx} \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}
\] (5-6)

• The equation above can also be obtained by writing the Taylor series expansion of the function \( f \) about the point \( x \),

\[
f(x + \Delta x) = f(x) + \Delta x \frac{df(x)}{dx} + \frac{1}{2} \Delta x^2 \frac{d^2 f(x)}{dx^2} + \ldots
\] (5-7)

and neglecting all the terms except the first two.
• The first term neglected is proportional to $\Delta x^2$, and thus the error involved in each step is also proportional to $\Delta x^2$.

• However, the *commutative error* involved after $M$ steps in the direction of length $L$ is proportional to $\Delta x$ since $M(\Delta x^2) = (L/\Delta x) \Delta x^2 = L \Delta x$.

\[ \text{Error} = L \Delta x \]
One-Dimensional Steady Heat Conduction

- Steady one-dimensional heat conduction in a plane wall of thickness $L$ with heat generation.

The wall is subdivided into $M$ sections of equal thickness $\Delta x = L/M$. 
• $M+1$ points $0, 1, 2, \ldots, m-1, m, m+1, \ldots, M$ called nodes or nodal points.

• The $x$-coordinate of any point $m$ is $x_m = m(\Delta x)$.

• The temperature at that point is simply $T(x_m) = T_m$.

• Internal ($1 \sim (M-1)$) and boundary nodal points ($0$ or $M$).
• Using Eq. 5–6

\[ \left. \frac{dT}{dx} \right|_{m-\frac{1}{2}} \approx \frac{T_m - T_{m-1}}{\Delta x} ; \quad \left. \frac{dT}{dx} \right|_{m+\frac{1}{2}} \approx \frac{T_{m+1} - T_m}{\Delta x} \] (5-8)

• Noting that the second derivative is simply the derivative of the first derivative:

\[ \left. \frac{d^2T}{dx^2} \right|_m \approx \frac{\left. \frac{dT}{dx} \right|_{m+\frac{1}{2}} - \left. \frac{dT}{dx} \right|_{m-\frac{1}{2}}}{\Delta x} = \frac{\frac{T_{m+1} - T_m}{\Delta x} - \frac{T_m - T_{m-1}}{\Delta x}}{\Delta x} \]

\[ = \frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} \] (5-9)
The governing equation for *steady one-dimensional* heat transfer in a plane wall with heat generation and constant thermal conductivity

\[
\frac{d^2 T}{dx^2} + \frac{\dot{e}}{k} = 0 \tag{5-10}
\]

\[
\left. \frac{d^2 T}{dx^2} \right|_m \approx \frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} \tag{5-9}
\]

\[
\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0, \quad m = 1, 2, 3, \ldots M - 1 \tag{5-11}
\]

Internal nodal points
-The equation is applicable to each of the $M-1$ interior nodes $\Rightarrow M-1$ equations for the determination of temperatures.

- The **two additional equations** needed to solve are obtained by applying the energy balance on the two elements at the boundaries.
Boundary Conditions

• A boundary node does not have a neighboring node on at least one side.

• *Energy balance* on the volume elements of boundary nodes is applied.

• Boundary conditions frequently encountered are:
  
  1. specified temperature,
  2. specified heat flux,
  3. convection, and
  4. radiation boundary conditions.
• Node number - at the left surface \((x=0)\): is 0,
- at the right surface at \((x=L)\): is \(M\)

• The width of the volume element: \(\Delta x/2\).

1. **Specified temperature boundary conditions:**

\[
T(0) = T_0 = \text{Specified value}
\]
\[
T(L) = T_M = \text{Specified value}
\]
• An energy balance on the volume element at that boundary:

\[ \sum_{All\ sides} \dot{Q} + \dot{E}_{gen,\text{element}} = 0 \quad (5-20) \]

• The finite difference formulation at the node \( m=0 \) can be expressed as:

\[
\dot{Q}_{\text{left surface}} + kA \frac{T_1 - T_0}{\Delta x} + \dot{e}_0 \left( A \Delta x / 2 \right) = 0 \quad (5-21)
\]

• The finite difference form of various boundary conditions can be obtained from Eq. 5–21 by replacing \( \dot{Q}_{\text{left surface}} \) by a suitable expression.
2. Specified Heat Flux Boundary Condition

\[ \dot{q}_0 A + kA \frac{T_1 - T_0}{\Delta x} + \dot{e}_0 \left( \frac{A\Delta x}{2} \right) = 0 \]  (5-22)

3. Convection Boundary Condition

\[ hA(T_\infty - T_0) + kA \frac{T_1 - T_0}{\Delta x} + \dot{e}_0 \left( \frac{A\Delta x}{2} \right) = 0 \]  (5-24)

4. Radiation Boundary Condition

\[ \varepsilon \sigma A(T_{\text{surr}}^4 - T_0^4) + kA \frac{T_1 - T_0}{\Delta x} + \dot{e}_0 \left( \frac{A\Delta x}{2} \right) = 0 \]  (5-25)

5. Combined Convection and Radiation

\[ hA(T_\infty - T_0) + \varepsilon \sigma A(T_{\text{surr}}^4 - T_0^4) + kA \frac{T_1 - T_0}{\Delta x} + \dot{e}_0 \left( \frac{A\Delta x}{2} \right) = 0 \]  (5-26)
The Mirror Image Concept

• The finite difference formulation of a node on an insulated boundary can be treated as “zero” heat flux is Eq. 5–23.

• Another and more practical way is to treat the node on an insulated boundary as an interior node.
• By replacing the insulation on the boundary by a mirror and considering the reflection of the medium as its extension

• Using Eq. 5.11:

\[
\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0
\]

\[
\frac{T_1 - 2T_0 + T_1}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0 \quad (5-30)
\]

\[
(T_1 - T_0) + \dot{e}_m \left( \frac{\Delta x^2}{2k} \right) = 0
\]
EXAMPLE 5–1  Steady Heat Conduction in a Large Uranium Plate

Consider a large uranium plate of thickness $L = 4$ cm and thermal conductivity $k = 28$ W/m·°C in which heat is generated uniformly at a constant rate of $\dot{e} = 5 \times 10^6$ W/m$^3$. One side of the plate is maintained at 0°C by iced water while the other side is subjected to convection to an environment at $T_\infty = 30$°C with a heat transfer coefficient of $h = 45$ W/m$^2$·°C, as shown in Figure 5–18. Considering a total of three equally spaced nodes in the medium, two at the boundaries and one at the middle, estimate the exposed surface temperature of the plate under steady conditions using the finite difference approach.

**FIGURE 5–18**
Schematic for Example 5–1.
FIGURE 5-18
Schematic for Example 5-1.

Uranium plate

\[ k = 28 \text{ W/m} \cdot ^\circ \text{C} \]
\[ \dot{e} = 5 \times 10^6 \text{ W/m}^3 \]
**SOLUTION**  A uranium plate is subjected to specified temperature on one side and convection on the other. The unknown surface temperature of the plate is to be determined numerically using three equally spaced nodes.

**Assumptions**  1 Heat transfer through the wall is steady since there is no indication of any change with time. 2 Heat transfer is one-dimensional since the plate is large relative to its thickness. 3 Thermal conductivity is constant. 4 Radiation heat transfer is negligible.

**Properties**  The thermal conductivity is given to be \( k = 28 \text{ W/m \cdot °C} \).

**Analysis**  The number of nodes is specified to be \( M = 3 \), and they are chosen to be at the two surfaces of the plate and the midpoint, as shown in the figure. Then the nodal spacing \( \Delta x \) becomes

\[
\Delta x = \frac{L}{M - 1} = \frac{0.04 \text{ m}}{3 - 1} = 0.02 \text{ m}
\]

We number the nodes 0, 1, and 2. The temperature at node 0 is given to be \( T_0 = 0 \text{ °C} \), and the temperatures at nodes 1 and 2 are to be determined. This problem involves only two unknown nodal temperatures, and thus we need to have only two equations to determine them uniquely. These equations are obtained by applying the finite difference method to nodes 1 and 2.
Node 1 is an interior node, and the finite difference formulation at that node is obtained directly from Eq. 5-18 by setting $m = 1$:

$$\frac{T_0 - 2T_1 + T_2}{\Delta x^2} + \frac{\dot{e}_1}{k} = 0 \rightarrow \frac{0 - 2T_1 + T_2}{\Delta x^2} + \frac{\dot{e}_1}{k} = 0 \rightarrow 2T_1 - T_2 = \frac{\dot{e}_1 \Delta x^2}{k}$$

(1)

Node 2 is a boundary node subjected to convection, and the finite difference formulation at that node is obtained by writing an energy balance on the volume element of thickness $\Delta x/2$ at that boundary by assuming heat transfer to be into the medium at all sides:

$$hA(T_\infty - T_2) + kA \frac{T_1 - T_2}{\Delta x} + \dot{e}_2(A\Delta x/2) = 0$$

Canceling the heat transfer area $A$ and rearranging give

$$T_1 - \left(1 + \frac{h\Delta x}{k}\right)T_2 = -\frac{h\Delta x}{k} T_\infty - \frac{\dot{e}_2 \Delta x^2}{2k}$$

(2)

Equations (1) and (2) form a system of two equations in two unknowns $T_1$ and $T_2$. Substituting the given quantities and simplifying gives

$$2T_1 - T_2 = 71.43 \quad \text{(in } ^\circ\text{C})$$
$$T_1 - 1.032T_2 = -36.68 \quad \text{(in } ^\circ\text{C})$$
This is a system of two algebraic equations in two unknowns and can be solved easily by the elimination method. Solving the first equation for \( T_1 \) and substituting into the second equation result in an equation in \( T_2 \) whose solution is

\[
T_2 = 136.1^\circ C
\]

This is the temperature of the surface exposed to convection, which is the desired result. Substitution of this result into the first equation gives \( T_1 = 103.8^\circ C \), which is the temperature at the middle of the plate.

**Discussion** The purpose of this example is to demonstrate the use of the finite difference method with minimal calculations, and the accuracy of the result was not a major concern. But you might still be wondering how accurate the result obtained above is. After all, we used a mesh of only three nodes for the entire plate, which seems to be rather crude. This problem can be solved analytically as described in Chapter 2, and the analytical (exact) solution can be shown to be

\[
T(x) = \frac{0.5\dot{e}hL^2/L + \dot{e}L + T_\infty h}{hL + k} x - \frac{\dot{e}x^2}{2k}
\]

Substituting the given quantities, the temperature of the exposed surface of the plate at \( x = L = 0.04 \) m is determined to be 136.0\(^\circ\)C, which is almost identical to the result obtained here with the approximate finite difference method (Fig. 5–19). Therefore, highly accurate results can be obtained with numerical methods by using a limited number of nodes.
Finite difference solution:

\[ T_2 = 136.1^\circ C \]

Exact solution:

\[ T_2 = 136.0^\circ C \]

**FIGURE 5–19**

Despite being approximate in nature, highly accurate results can be obtained by numerical methods.
Homework

10 segments are selected (11 node points)

EES Answer:
T1=26.46°C, T2=50.06°C, T3=70.81°C, T4=88.7°C, T5=103.7°C,
T6=115.6°C, T7=125.2°C, T8=131.7°C, T9=135.3°C, T10=136°C